Algorithms and Data Structures

Lec03

Algorithm Analysis

Dr. Mohammad Ahmad

Algorithm Analysis

Outline:

In this topic, we will examine code to determine the run time of various operations.

We will calculate the run times of:

– Operators **+**, **-**, **=**, **+=**, **++**, etc.

– Control statements **`if`**, **`for`**, **`while`**, **`do-while`**, **`switch`**

– Functions

– Recursive functions

Motivation

The goal of algorithm analysis is to take a block of code and determine the asymptotic run time or asymptotic memory requirements based on various parameters

- Given an array of size n:
  - Selection sort requires $\Theta(n^2)$ time
  - Merge sort, quick sort, and heap sort all require $\Theta(n \ln(n))$ time
- However:
  - Merge sort requires $\Theta(n)$ additional memory
  - Quick sort requires $\Theta(\ln(n))$ additional memory
  - Heap sort requires $\Theta(1)$ memory

Motivation

To properly investigate the determination of run times asymptotically:

- We will begin with machine instructions
- Discuss operations
- Control statements
    - Conditional statements and loops
- Functions
- Recursive functions

Operators

Because each machine instruction can be executed in a fixed number of cycles, we may assume each operation requires a fixed number of cycles

- The time required for any operator is $\Theta(1)$ including:
    - Retrieving/storing variables from memory
    - Variable assignment **=**
    - Integer operations **+ - * / % ++ --**
    - Logical operations **&& || !**
    - Bitwise operations **& | ^ ~**
    - Relational operations **== != < <= => >**
    - Memory allocation and deallocation **new delete**

Blocks of Operations

Each operation runs in $\Theta(1)$ time and therefore any fixed number of operations also run in $\Theta(1)$ time, for example:

```
// Swap variables a and b
int tmp = a;
a = b;
b = tmp;
```

Blocks in Sequence

Suppose you have now analyzed a number of blocks of code run in sequence

```
template <typename T>
void update_capacity( int delta ) {
    T *array_old = array;
    int capacity_old = array_capacity;
    array_capacity += delta;
    array = new T[array_capacity];

    for ( int i = 0; i < capacity_old; ++i ) {
            array[i] = array_old[i];
    }

    delete[] array_old;
}
```

$\Theta(1)$

$\Theta(n)$

$\Theta(1)$

To calculate the total run time, add the entries: $\Theta(1 + n + 1) = \Theta(n)$

Blocks in Sequence

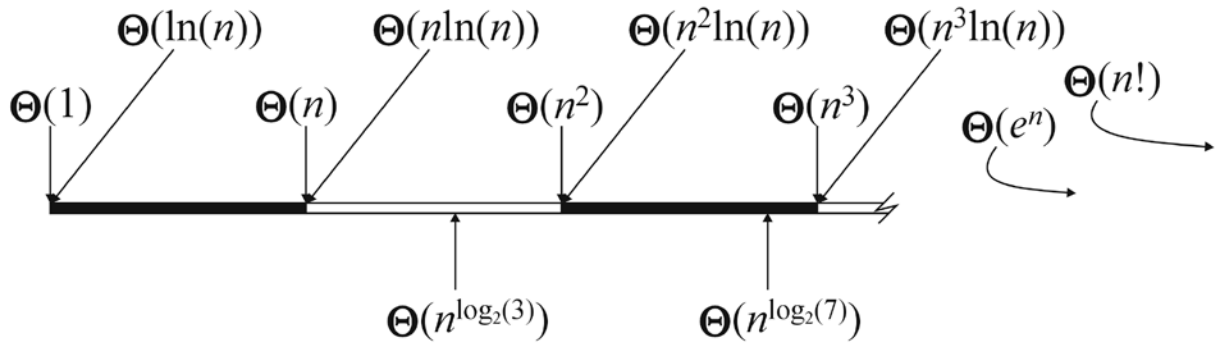
Other examples include:

- Run three blocks of code which are $\Theta(1)$, $\Theta(n^2)$, and $\Theta(n)$

  Total run time $\Theta(1 + n^2 + n) = \Theta(n^2)$

- Run two blocks of code which are $\Theta(n \ln(n))$, and $\Theta(n^{1.5})$

  Total run time $\Theta(n \ln(n) + n^{1.5}) = \Theta(n^{1.5})$



- When considering a sum, take the dominant term

Control Statements

Next we will look at the following control statements

These are statements which potentially alter the execution of instructions

- Conditional statements
  - **`if`**, **`switch`**
- Condition-controlled loops
  - **`for`**, **`while`**, **`do-while`**
- Count-controlled loops
  - **`for i from 1 to 10 do ... end do;`**
- Collection-controlled loops
  - **`foreach ( int i in array ) { ... }      // C#`**

Control Statements

Given any collection of nested control statements, it is always necessary to work inside out

– Determine the run times of the inner-most statements and work your way out

Control Statements

Given

```
if ( condition ) {
    // true body
} else {
    // false body
}
```

The run time of a conditional statement is:

- the run time of the condition (the test), plus
- the run time of the body which is run

In most cases, the run time of the condition is $\Theta(1)$

Control Statements

In some cases, it is easy to determine which statement must be run:

```
int factorial ( int n ) {
        if ( n == 0 ) {
                return 1;
        } else {
                return n * factorial ( n – 1 );
        }
}
```

Control Statements

In others, it is less obvious

– Find the maximum entry in an array:

```
int find_max( int *array, int n ) {
    max = array[0];

    for ( int i = 1; i < n; ++i ) {
        if ( array[i] > max ) {
            max = array[i];
        }
    }

    return max;
}
```

Condition-controlled Loops

The for loop is a condition controlled statement:

```
for ( int i = 0; i < N; ++i ) {
        // ...
}
```

is identical to

```
int i = 0;                              // initialization
while ( i < N ) {                       // condition
    // ...
    ++i;                                // increment
}
```

Condition-controlled Loops

The initialization, condition, and increment usually are single statements running in $\Theta(1)$

```
for ( int i = 0; i < N; ++i ) {
        // ...
}
```

Condition-controlled Loops

The initialization, condition, and increment statements are usually $\Theta(1)$

For example,

```
for ( int i = 0; i < n; ++i ) {
    // ...
}
```

Assuming there are no break or return statements in the loop,
the run time is $\Omega(n)$

Condition-controlled Loops

If the body does not depend on the variable (in this example, `i`), then the run time of

```
for ( int i = 0; i < n; ++i ) {
    // code which is Theta(f(n))
}
```

is $\mathbf{\Theta}(n\,\mathrm{f}(n))$

If the body is $\mathbf{O}(\mathrm{f}(n))$, then the run time of the loop is $\mathbf{O}(n\,\mathrm{f}(n))$

Condition-controlled Loops

For example,

```
int sum = 0;
for ( int i = 0; i < n; ++i ) {
    sum += 1;   //  Theta(1)
}
```

This code has run time

$$\Theta(n \cdot 1) = \Theta(n)$$

Condition-controlled Loops

Another example example,

```
int sum = 0;
for ( int i = 0; i < n; ++i ) {
    for ( int j = 0; j < n; ++j ) {
        sum += 1;     Theta(1)
    }
}
```

The previous example showed that the inner loop is $\Theta(n)$, thus the outer loop is

$$\Theta(n \cdot n) = \Theta(n^2)$$

Analysis of Repetition Statements

Suppose with each loop, we use a linear search an array of size m:

```
for ( int i = 0; i < n; ++i ) {
      // search through an array of size m
      // O( m );
}
```

The inner loop is $\mathbf{O}(m)$ and thus the outer loop is

$$\mathbf{O}(n\ m)$$

Conditional Statements

Consider this example

```
void Disjoint_sets::clear() {
    if ( sets == n ) {
        return;
    }

    max_height = 0;
    num_disjoint_sets = n;

    for ( int i = 0; i < n; ++i ) {
        parent[i] = i;
        tree_height[i] = 0;
    }
}
```

$\Theta(1)$

$\Theta(1)$

$\Theta(n)$

$\Theta(1)$

$$T_{\text{clear}}(n) = \begin{cases} \Theta(1) & sets = n \\ \Theta(n) & \text{otherwise} \end{cases}$$

Analysis of Repetition Statements

If the body does depends on the variable (in this example, `i`), then the run time of

```
for ( int i = 0; i < n; ++i ) {
    // code which is Theta(f(i,n))
}
```

is $\Theta\left(1 + \sum_{i=0}^{n-1} 1 + \mathrm{f}(i,n)\right)$ and if the body is

$\mathbf{O}(\mathrm{f}(i, n))$, the result is

$$\mathbf{O}\left(1 + \sum_{i=0}^{n-1} 1 + \mathrm{f}(i,n)\right)$$

Analysis of Repetition Statements

For example,

```
int sum = 0;
for ( int i = 0; i < n; ++i ) {
    for ( int j = 0; j < i; ++j ) {
        sum += i + j;
    }
}
```

The inner loop is $\mathbf{O}(1 + i(1 + 1)) = \mathbf{\Theta}(i)$ hence the outer is

$$\mathbf{\Theta}\left(1 + \sum_{i=0}^{n-1} 1 + i\right) = \mathbf{\Theta}\left(1 + n + \sum_{i=0}^{n-1} i\right) = \mathbf{\Theta}\left(1 + n + \frac{n(n-1)}{2}\right) = \mathbf{\Theta}\left(n^2\right)$$

Analysis of Repetition Statements

As another example:

```
int sum = 0;
for ( int i = 0; i < n; ++i ) {
    for ( int j = 0; j < i; ++j ) {
        for ( int k = 0; k < j; ++k ) {
            sum += i + j + k;
        }
    }
}
```

From inside to out:

$\Theta(1)$

$\Theta(j)$

$\Theta(i^2)$

$\Theta(n^3)$

Control Statements

Switch statements appear to be nested if statements:

```
switch( i ) {
     case 1:    /* do stuff */ break;
     case 2:    /* do other stuff */ break;
     case 3:    /* do even more stuff */ break;
     case 4:    /* well, do stuff */ break;
     case 5:    /* tired yet? */ break;
     default:   /* do default stuff */
}
```

Control Statements

Thus, a switch statement would appear to run in $\mathbf{O}(n)$ time where n is the number of cases, the same as nested if statements

– Why then not use:

```
if ( i == 1 ) { /* do stuff */ }
else if ( i == 2 ) { /* do other stuff */ }
else if ( i == 3 ) { /* do even more stuff */ }
else if ( i == 4 ) { /* well, do stuff */ }
else if ( i == 5 ) { /* tired yet? */ }
else { /* do default stuff */ }
```

Serial Statements

Suppose we run one block of code followed by another block of code

Such code is said to be run *serially*

If the first block of code is $\mathbf{O}(f(n))$ and the second is $\mathbf{O}(g(n))$, then the run time of two blocks of code is

$$\mathbf{O}(\ f(n) + g(n)\ )$$

which usually (for algorithms not including function calls) simplifies to one or the other

Serial Statements

Consider the following two problems:

- search through a random list of size n to find the maximum entry, and
- search through a random list of size n to find if it contains a particular entry

What is the proper means of describing the run time of these two algorithms?

Serial Statements

Searching for the maximum entry requires that each element in the array be examined, thus, it must run in $\mathbf{\Theta}(n)$ time

Searching for a particular entry may end earlier: for example, the first entry we are searching for may be the one we are looking for, thus, it runs in $\mathbf{O}(n)$ time

Serial Statements

Therefore:

- if the leading term is big-$\mathbf{\Theta}$, then the result must be big-$\mathbf{\Theta}$, otherwise
- if the leading term is big-$\mathbf{O}$, we can say the result is big-$\mathbf{O}$

For example,

$$\mathbf{O}(n) + \mathbf{O}(n^2) + \mathbf{O}(n^4) = \mathbf{O}(n + n^2 + n^4) = \mathbf{O}(n^4)$$

$$\mathbf{O}(n) + \mathbf{\Theta}(n^2) = \mathbf{\Theta}(n^2)$$

$$\mathbf{O}(n^2) + \mathbf{\Theta}(n) = \mathbf{O}(n^2)$$

$$\mathbf{O}(n^2) + \mathbf{\Theta}(n^2) = \mathbf{\Theta}(n^2)$$

Functions

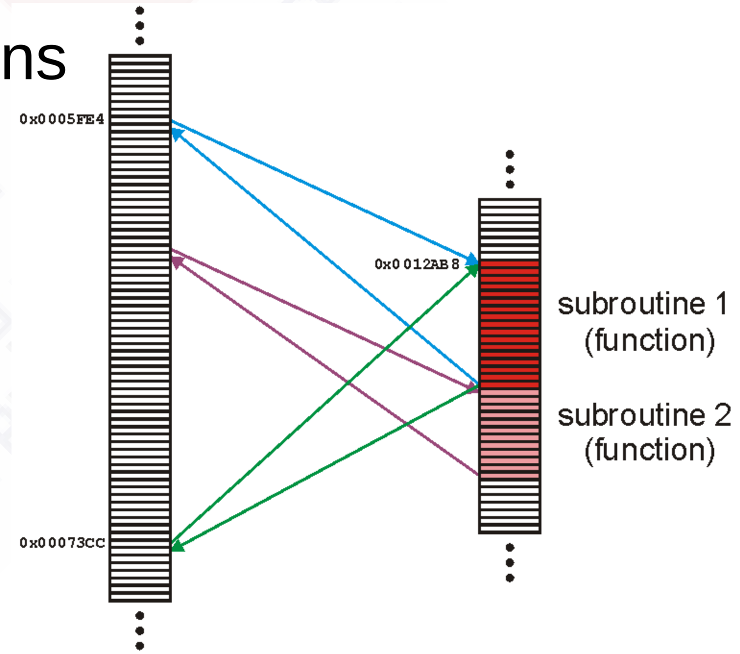
A function (or subroutine) is code which has been separated out, either to:

– and repeated operations

  • e.g., mathematical functions

– group related tasks

  • e.g., initialization



Functions

Because a subroutine (function) can be called from anywhere, we must:

- prepare the appropriate environment
- deal with arguments (parameters)
- jump to the subroutine
- execute the subroutine
- deal with the return value
- clean up

Functions

Fortunately, this is such a common task that all modern processors have instructions that perform most of these steps in one instruction

Thus, we will assume that the overhead required to make a function call and to return is $\Theta(1)$

Functions

Because any function requires the overhead of a function call and return, we will always assume that

$$T_f = \Omega(1)$$

That is, it is impossible for any function call to have a zero run time

Functions

Thus, given a function $f(n)$ (the run time of which depends on n) we will associate the run time of $f(n)$ by some function $T_f(n)$

– We may write this to $T(n)$

Because the run time of any function is at least $\mathbf{O}(1)$, we will include the time required to both call and return from the function in the run time

Functions

Consider this function:

```
void Disjoint_sets::set_union( int m, int n ) {
        m = find( m );
        n = find( n );

        if ( m == n ) {
                return;
        }

        --num_disjoint_sets;

        if ( tree_height[m] >= tree_height[n] ) {
            parent[n] = m;

            if ( tree_height[m] == tree_height[n] ) {
                ++( tree_height[m] );
                max_height = std::max( max_height, tree_height[m] );
            }
        } else {
            parent[m] = n;
        }
}
```

$2T_{find}$

$T_{set\_union} = 2T_{find} + \Theta(1)$

$\Theta(1)$

Recursive Functions

A function is relatively simple (and boring) if it simply performs operations and calls other functions

Most interesting functions designed to solve problems usually end up calling themselves

– Such a function is said to be *recursive*

Recursive Functions

As an example, we could implement the factorial function recursively:

```
int factorial( int n ) {
    if ( n <= 1 ) {
        return 1;
    } else {
        return n * factorial( n - 1 );
    }
}
```

$\Theta(1)$

$\mathrm{T}_!(n-1)+\Theta(1)$

Recursive Functions

Thus, we may analyze the run time of this function as follows:

$$\mathrm{T}_{!}(n) = \begin{cases} \mathbf{\Theta}(1) & n \le 1 \\ \mathrm{T}_{!}(n-1) + \mathbf{\Theta}(1) & n > 1 \end{cases}$$

We don’t have to worry about the time of the conditional ($\mathbf{\Theta}(1)$) nor is there a probability involved with the conditional statement

Recursive Functions

The analysis of the run time of this function yields a recurrence relation:

$$T_!(n) = T_!(n-1) + \Theta(1) \qquad T_!(1) = \Theta(1)$$

This recurrence relation has Landau symbols…

– Replace each Landau symbol with a representative function:

$$T_!(n) = T_!(n-1) + 1 \qquad T_!(1) = 1$$

Recursive Functions

Thus, to find the run time of the factorial function, we need to solve

$$\mathrm{T}_!(n) = \mathrm{T}_!(n-1) + 1 \qquad \mathrm{T}_!(1) = 1$$

```
solve {T(n) = T(n – 1) + 1, T(1) = 1}
```

n

Thus, $\mathrm{T}_!(n) = \Theta(n)$

Recursive Functions

We can examine the first few steps:

$$
\begin{aligned}
\mathrm{T}_!(n) \quad &= \mathrm{T}_!(n-1) + 1 \\
&= \mathrm{T}_!(n-2) + 1 + 1 = \mathrm{T}_!(n-2) + 2 \\
&= \mathrm{T}_!(n-3) + 3
\end{aligned}
$$

From this, we see a pattern:

$$\mathrm{T}_!(n) = \mathrm{T}_!(n-k) + k$$

Recursive Functions

If $k = n - 1$ then

$$\begin{aligned} \mathrm{T}_!(n) &= \mathrm{T}_!(n - (n - 1)) + n - 1 \\ &= \mathrm{T}_!(1) + n - 1 \\ &= 1 + n - 1 = n \end{aligned}$$

Thus, $\mathrm{T}_!(n) = \Theta(n)$

Recursive Functions

Analyzing the function, we get:

```
void sort( int * array, int n ) {
    if ( n <= 1 ) {
        return;
    }

    int posn = 0;
    int max = array[posn];

    for ( int i = 1; i < n; ++i ) {
        if ( array[i] > max ) {
            posn = i;
            max = array[posn];
        }
    }

    int tmp = array[n - 1];
    array[n - 1] = array[posn];
    array[posn] = tmp;

    sort( array, n - 1 );
}
```

$\mathrm{T}(0) = \mathrm{T}(1) = \Theta(1)$

$\Theta(1)$

$\Theta(1)$

$\Theta(n)$

$\Theta(1)$

$\mathrm{T}(n-1)$

$$\begin{aligned}\mathrm{T}(n) &= \Theta(1) + \Theta(n) + \Theta(1) + \mathrm{T}(n-1)\\ &= \mathrm{T}(n-1) + \Theta(n)\end{aligned}$$

Recursive Functions

Thus, replacing each Landau symbol with a representative, we are required to solve the recurrence relation

$$\mathrm{T}(n) = \mathrm{T}(n-1) + n \qquad \mathrm{T}(1) = 1$$

$$-1 - n + (n+1)\left(\frac{n}{2} + 1\right)$$

$$\frac{1}{2}n + \frac{1}{2}n^2$$

Recursive Functions

Consequently, the sorting routine has the run time

$$T(n) = \Theta(n^2)$$

To see this by hand, consider the following

$$\begin{aligned}
T(n) &= T(n-1) + n \\
&= \big(T(n-2) + (n-1)\big) + n \\
&= T(n-2) + n + (n-1) \\
&= T(n-3) + n + (n-1) + (n-2) \\
&\vdots \\
&= T(1) + \sum_{i=2}^{n} i = 1 + \sum_{i=2}^{n} i = \sum_{i=1}^{n} i = \frac{n(n+1)}{2}
\end{aligned}$$

Recursive Functions

Consider, instead, a binary search of a sorted list:

- Check the middle entry
- If we do not find it, check either the left- or right-hand side, as appropriate

Thus, $\mathrm{T}(n) = \mathrm{T}((n - 1)/2) + \Theta(1)$

Recursive Functions

Also, if $n = 1$, then $\mathrm{T}(1) = \Theta(1)$

Thus we have to solve:

$$\mathrm{T}(n) = \begin{cases} 1 & n = 1 \\ \mathrm{T}\left(\frac{n-1}{2}\right) + 1 & n > 1 \end{cases}$$

Solving this can be difficult, in general, so we will consider only special values of n

Assume $n = 2^k - 1$ where k is an integer

Then $(n - 1)/2 = (2^k - 1 - 1)/2 = 2^{k-1} - 1$

Recursive Functions

For example, searching a list of size 31 requires us to check the center

If it is not found, we must check one of the two halves, each of which is size 15

$$31 = 2^5 - 1$$

$$15 = 2^4 - 1$$

Recursive Functions

Thus, we can write

$$
\begin{aligned}
\mathrm{T}(n) &= \mathrm{T}(2^k - 1) \\
&= \mathrm{T}\left(\frac{2^k - 1 - 1}{2}\right) + 1 \\
&= \mathrm{T}(2^{k-1} - 1) + 1 \\
&= \mathrm{T}\left(\frac{2^{k-1} - 1 - 1}{2}\right) + 1 + 1 \\
&= \mathrm{T}(2^{k-2} - 1) + 2 \\
&\vdots
\end{aligned}
$$

Recursive Functions

Notice the pattern with one more step:

$$
\begin{aligned}
&= T(2^{k-1}-1)+1 \\
&= T\left(\frac{2^{k-1}-1-1}{2}\right)+1+1 \\
&= T(2^{k-2}-1)+2 \\
&= T(2^{k-3}-1)+3 \\
&\vdots
\end{aligned}
$$

Recursive Functions

Thus, in general, we may deduce that after $k-1$ steps:

$$\begin{aligned} \mathrm{T}(n) &= \mathrm{T}(2^{k}-1) \\ &= \mathrm{T}(2^{k-(k-1)}-1)+k-1 \\ &= \mathrm{T}(1)+k-1=k \end{aligned}$$

because $\mathrm{T}(1) = 1$

Recursive Functions

Thus, $T(n) = k$, but $n = 2^k - 1$
Therefore $k = \lg(n + 1)$

However, recall that $f(n) = \Theta(g(n))$ if $\lim_{n\to\infty} \frac{f(n)}{g(n)} = c$
for $0 < c < \infty$

$$\lim_{n\to\infty} \frac{\lg(n+1)}{\ln(n)} = \lim_{n\to\infty} \frac{\frac{1}{(n+1)\ln(2)}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{(n+1)\ln(2)} = \frac{1}{\ln(2)}$$

Thus, $T(n) = \Theta(\lg(n + 1)) = \Theta(\ln(n))$

Cases

As well as determining the run time of an algorithm, because the data may not be deterministic, we may be interested in:

– Best-case run time

– Average-case run time

– Worst-case run time

In many cases, these will be significantly different

Cases

Searching a list linearly is simple enough

We will count the number of comparisons

- Best case:
  - The first element is the one we're looking for: $\mathbf{O}(1)$
- Worst case:
  - The last element is the one we're looking for, or it is not in the list: $\mathbf{O}(n)$
- Average case?
  - We need some information about the list...

Cases

Assume the case we are looking for is in the list and equally likely distributed

If the list is of size n, then there is a $1/n$ chance of it being in the ith location

Thus, we sum

$$\frac{1}{n}\sum_{i=1}^{n} i = \frac{1}{n}\frac{n(n+1)}{2} = \frac{n+1}{2}$$

which is $\mathbf{O}(n)$